ISRAEL JOURNAL OF MATHEMATICS, Vol. 73, No. 3, 1991

ON THE SINGULARITIES OF NILPOTENT ORBITS

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ABSTRACT

Let $\mathcal{O}$ be a nilpotent orbit of the adjoint action of a complex connected semi-simple Lie group on its Lie algebra. We prove that the normalization of the closure of $\mathcal{O}$ is Gorenstein and has rational singularities.

1. Introduction

Let G be a complex connected semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. Let $e \in \mathfrak{g}$ be a nilpotent element. It is well known that the orbit $\mathcal{O} = G \cdot e \subseteq \mathfrak{g}$ of e under the adjoint action of G on $\mathfrak{g}$ is locally closed in the Zariski topology, that is, $\mathcal{O}$ is open in its Zariski closure $\bar{\mathcal{O}}$. The latter is an affine algebraic variety defined by the ideal of polynomials in $\mathbf{C}[\mathfrak{g}] = S(\mathfrak{g}^*)$ vanishing on $\mathcal{O}$.

The aim of this paper is to prove that the normalization $\bar{\mathcal{O}}^{\text{norm}}$ of $\bar{\mathcal{O}}$ is Gorenstein and has rational singularities (for the definitions see (2.1)–(2.2)). We deduce from this that the singularity of $\bar{\mathcal{O}}^{\text{norm}}$ in a G-orbit of codimension two is smoothly equivalent to a rational double point.

Interest in the study of singularities of nilpotent orbits comes from Brieskorn's work [B]. There he proves that the singularity of the nilpotent cone of a complex semi-simple Lie algebra in a subregular point is always smoothly equivalent to a rational double point; in the cases A_n, D_n, E_6–E_8 this singularity is described by the same Dynkin diagram as the one corresponding to the Lie algebra.

It has been proven for many special cases that the normalization of a nilpotent orbit has rational singularities (cf. [He1], [KP], [K], [Kr]). However, the general result seemed to be difficult.

The starting point for our considerations was W. McGovern's formula (cf. [M])

Received November 23, 1990

expressing the G-module structure of the ring of global regular functions on $\mathcal{O}$ in terms of some induced modules.

The proof is a direct application of duality theory for coherent sheaves (see [H], Ch. VII).

The results and constructions of [M] are crucial in our proof. Since the original paper contains some misprints and inaccuracies, we present in Sections 3.1–3.2 an outline of the constructions we need. In this I would like to thank W. McGovern for making clear to me some aspects of his proof.

I would like to express my deep gratitude to Professor A. Joseph for valuable comments and suggestions, to V. Berkovich for helpful discussions.

The paper was written in January 1990. It was available as a preprint since February 2, 1990. The main result, Theorem (3.3), was incorporated into a Ph.D. thesis by A. Broer ([Bro], 3.6.19, 4.2.11) who informed us in a letter of May 25, 1990 that he had previously known this result in a special case. On July 3, 1990 we received from H. Kraft a handwritten manuscript by D. Panyushev giving a rather similar proof of Theorem (3.3) — see Section 6.

2. The key lemma

Throughout the paper, all schemes are assumed to be schemes of finite type over **C**.

Let us recall a few definitions.

(2.1) Definition. (a) A commutative Noetherian ring is called *Gorenstein* if it has finite injective dimension as a module over itself (for equivalent definitions see [Ma], Th. 18.1).

(b) A scheme S is *Gorenstein* if all the local rings $\mathcal{O}_{S,x}$, $x \in S$, are Gorenstein.

(2.2) Definition (cf. [KKMS], pp. 50–51). A scheme S is said to have *rational singularities* if it is normal and there exists a desingularization $f: X \to S$ such that $R^i f_*(\mathcal{O}_X) = 0$ for $i > 0$.

A key step in the proof of our main result is the following lemma.

(2.3) Lemma. *Let $\pi: X \to Y$ be a proper birational morphism with X smooth and Y normal. Let ω_X be the sheaf of higher differentials on X. Suppose there exists a morphism $\phi: \mathcal{O}_X \to \omega_X$ such that $\pi_*\phi: \pi_*\mathcal{O}_X \to \pi_*\omega_X$ is an isomorphism. Then Y is Gorenstein and has rational singularities.*

The proof of the lemma is given in Section 5. A sketch of the duality theorem needed is presented in Section 4.

3. Applications to nilpotent orbits

In 3.1–3.2 we present an outline of McGovern's results [M]. Sections 3.3–3.4 contain the main results of this paper.

Let G, $\mathfrak{g}$, e, $\mathcal{O}$ be as in the Introduction.

(3.1) Desingularization of $\bar{\mathcal{O}}$ (cf. [KP], [M])

Choose an $\mathfrak{sl}_2$-triple (e,f,h) in $\mathfrak{g}$ containing the nilpotent element $e \in \mathfrak{g}$. Set $\mathfrak{g}_i = \{x \in \mathfrak{g} : [h,x] = ix\}$ for $i \in \mathbf{Z}$. One has $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}_i$. Moreover, $\mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}_i$ is a parabolic subalgebra of $\mathfrak{g}$ having nil-radical $\mathfrak{m} := \bigoplus_{i \geq 1} \mathfrak{g}_i$ and Levi subalgebra $\mathfrak{l} := \mathfrak{g}_0$. Denote by V the subalgebra $\bigoplus_{i \geq 2} \mathfrak{g}_i$. Obviously $e \in \mathfrak{g}_2 \subseteq V$.

Let P be the parabolic subgroup of G with Lie algebra $\mathfrak{p} \subseteq \mathfrak{g}$, and let $P = LM$ be the Levi decomposition for P corresponding to $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$.

(3.1.1) Lemma ([SS], III.4.16–4.19). (a) $\mathrm{Stab}_G(e) \subseteq P$;
(b) $V = \overline{Pe}$.

From this a desingularization for $\bar{\mathcal{O}}$ can be constructed as follows. Define $X = G \times^P V = (G \times V)/P$, where the right action of P on $G \times V$ is defined by the formula $(g,v)p = (gp, p^{-1}v)$. Define the map $\tau : X \to \mathfrak{g}$ by the formula $\tau(g,v) = g(v) \in \mathfrak{g}$. Since τ is a composition of the closed embedding $X \hookrightarrow G/P \times \mathfrak{g}$, $(g,v) \mapsto (gP, g(v))$, and the projection $\mathrm{pr}_2 : G/P \times \mathfrak{g} \to \mathfrak{g}$, it is proper.

(3.1.2) Proposition ([KP], 7.4; [M]). *The image $\tau(X)$ is $\bar{\mathcal{O}} \subseteq \mathfrak{g}$ and $\tau : X \to \bar{\mathcal{O}}$ is proper birational. It is an isomorphism on $\tau^{-1}(\mathcal{O}) \subseteq X$.*

(3.2) McGovern's formula [M]

The space X constructed above admits a natural G-action. This provides a natural G-action on the structure sheaf $\mathcal{O}_X$ and also on the sheaf ω_X of higher differentials.

Consider the following exact sequence of P-modules

$$0 \to V \to \mathfrak{m} \to \mathfrak{g}_1 \to 0$$

where $\mathfrak{g}_1$ is extended to a P-module by letting M act trivially. Its dual

$$0 \to \mathfrak{g}_1^* \to \mathfrak{m}^* \to V^* \to 0$$

gives rise to the following Koszul-type resolution for a $P - S(\mathfrak{m}^*)$-module $S(V^*)$

$$(1) \qquad 0 \to \bigwedge^t \mathfrak{g}_1^* \otimes S(\mathfrak{m}^*) \to \cdots \to \mathfrak{g}_1^* \otimes S(\mathfrak{m}^*) \to S(\mathfrak{m}^*) \to S(V^*) \to 0$$

where $t = \dim_{\mathbf{C}} \mathfrak{g}_1$ and $S(\ )$ denotes the symmetric algebra.

Consider the following diagram of schemes.

$$\begin{array}{ccc} X = G \times^P V & \xrightarrow{i} & G \times^P \mathfrak{m} =: Y \\ & \searrow^{p} \quad {}^{q}\swarrow & \\ & G/P & \end{array}$$

The morphism $q: Y \to G/P$ is affine. It defines an equivalence q_* between the category of $\mathcal{O}_Y$-modules and the category of $q_*(\mathcal{O}_Y)$-modules. Moreover, q_* also defines an equivalence between the category of $G - \mathcal{O}_Y$-modules and the category of $G - q_*(\mathcal{O}_Y)$-modules.

For any finite-dimensional P-module W let $\widetilde{W}$ denote the G-sheaf of sections of the vector bundle $G \times^P W$ over G/P. The correspondence $W \mapsto \widetilde{W}$ is an exact functor; it induces an exact functor from the category of $P - S(\mathfrak{m}^*)$-modules of finite type to the category of $G - S(\widetilde{\mathfrak{m}}^*)$-modules.

Note that $q_*(\mathcal{O}_Y) = S(\widetilde{\mathfrak{m}}^*)$, so, composing the functors above, we obtain an exact functor from the category of finitely generated $P - S(\mathfrak{m}^*)$-modules into the category of $G - \mathcal{O}_Y$-modules. Applying this functor to (1) one obtains the following G-equivariant resolution for the $\mathcal{O}_Y$-module $i_*\mathcal{O}_X$:

$$0 \to q^*\left(\bigwedge^t \tilde{\mathfrak{g}}_1^*\right) \to \cdots \to q^*(\tilde{\mathfrak{g}}_1^*) \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0. \tag{2}$$

In a similar way one obtains the following G-equivariant resolution for the $\mathcal{O}_Y$-module $i_*\omega_X$:

$$0 \to \mathcal{O}_Y \to q^*(\tilde{\mathfrak{g}}_1) \to \cdots \to q^*\left(\bigwedge^t \tilde{\mathfrak{g}}_1\right) \to i_*\omega_X \to 0. \tag{3}$$

The resolution (3) leads easily to the following formula for the Euler characteristic of ω_X as an element of the representation ring of G:

$$\sum_i (-1)^i H^i(X, \omega_X) = \sum_{i,j} (-1)^{i+j} H^i\left(G/P, q_*\mathcal{O}_Y \otimes \bigwedge^j \tilde{\mathfrak{g}}_1\right). \tag{4}$$

According to the Grauert–Riemenschneider theorem (cf. [GR], Satz 2.4 or [K], Theorem 4), $H^i(X, \omega_X) = 0$ for $i > 0$.

On the other hand, a nice lemma ([M], Lemma 2.1) expressing induction functors Ind_L^G in terms of cohomology of G/P, makes it possible to rewrite the right-hand side of (4) and to obtain the following

(3.2.1) PROPOSITION. *One has*

$$H^0(X,\omega_X) = \sum_j (-1)^j \operatorname{Ind}_L^G\left(\bigwedge^j \mathfrak{g}_1\right)$$

in the representation ring of G.

The following result is of great importance for us.

(3.2.2) PROPOSITION. *There exists a G-invariant section in $H^0(X,\omega_X)$. Let $\phi: \mathcal{O}_X \to \omega_X$ be the corresponding sheaf morphism. Then*

$$H^0(X,\phi): H^0(X,\mathcal{O}_X) \to H^0(X,\omega_X)$$

is bijective.

PROOF. The existence of a G-invariant section follows from the formula (3.2.1). Indeed, by Frobenius reciprocity the trivial representation of G appears only once in the right-hand side, namely, in the summand $\operatorname{Ind}_L^G(\bigwedge^0 \mathfrak{g}_1)$. Denote by $\phi: \mathcal{O}_X \to \omega_X$ the corresponding G-equivariant morphism of sheaves. We wish to prove that $H^0(X,\phi): H^0(X,\mathcal{O}_X) \to H^0(X,\omega_X)$ is bijective.

Injectivity of $H^0(X,\phi)$ is standard. To prove surjectivity, note that any homomorphism $\psi: \mathcal{O}_X \to \omega_X$ can be represented as $\psi = \phi \cdot x$ where x is a rational function on X. Consider x as a rational function on $Y = \bar{\mathcal{O}}^{\text{norm}}$. The set Ω of poles of x is a subset of the set Ω' of zeroes of ϕ. But Ω' is a G-invariant proper subset of Y. Hence $\operatorname{codim}\Omega \geq \operatorname{codim}\Omega' \geq 2$. Thus x is regular on $\bar{\mathcal{O}}^{\text{norm}}$. This means that the map $H^0(Y,\mathcal{O}_Y) \to H^0(X,\omega_X)$, defined by the G-invariant section of ω_X, is onto. Since the morphism $X \to Y$ is birational and proper, and Y is normal, $H^0(X,\mathcal{O}_X) = H^0(Y,\mathcal{O}_Y)$, and this finishes the proof.

(3.2.3) COROLLARY ([M], Theorem 3.1). *Let $R(\mathcal{O})$ be the ring of regular functions on $\mathcal{O}$ considered as a G-module.*

One has

$$R(\mathcal{O}) = \sum_j (-1)^j \operatorname{Ind}_L^G\left(\bigwedge^j \mathfrak{g}_1\right)$$

in the representation ring of G.

(3.3) Singularities of $\bar{\mathcal{O}}^{\text{norm}}$

Set $Y = \bar{\mathcal{O}}^{\text{norm}}$. The morphism $\tau: X \to \bar{\mathcal{O}}$ described in (3.1.2) can be factored through normalization $Y \to \bar{\mathcal{O}}$. Denote by π the morphism from X to Y so obtained. Since Y is affine and $H^0(X,\phi)$ is an isomorphism by (3.2.2), it follows that $\pi_*\phi : \pi_*\mathcal{O}_X \to \pi_*\omega_X$ is also an isomorphism. Since $\bar{\mathcal{O}}$ is finitely generated over $\mathbf{C}$ so is Y. Applying Lemma (2.3) to $\pi : X \to Y$ we obtain immediately the result announced:

THEOREM. *Let G be a complex connected semi-simple Lie group with the Lie algebra $\mathfrak{g}$. Let $\mathcal{O}$ be a nilpotent orbit of the adjoint action of G on $\mathfrak{g}$. Then the normalization $\bar{\mathcal{O}}^{\mathrm{norm}}$ of the closure $\bar{\mathcal{O}}$ is Gorenstein and has rational singularities.*

(3.4) Singularities in codimension two

Theorem (3.3) gives particularly nice information about the singularities of $\bar{\mathcal{O}}^{\mathrm{norm}}$ in codimension two.

Recall the following definitions.

(3.4.1) DEFINITION (cf. [He2], 1.7 or [KP], 12.1). Algebraic varieties X and Y are said to be *smoothly equivalent* in $x \in X$ and $y \in Y$ if there exist a variety Z, a point $z \in Z$ and smooth morphisms $f : Z \to X$, $g : Z \to Y$, such that $f(z) = x$, $g(z) = y$.

(3.4.2) DEFINITION (cf. [S], III.5.1). Let X be a variety with an action of an algebraic group G. Take $x \in X$. A locally closed subvariety $S \ni x$ is called a *transverse slice* in X to the G-orbit of x if

(i) the morphism $\mu : (g,s) \mapsto gs$ of $G \times S$ into X is smooth,

(ii) the dimension of S is minimal with respect to (i).

(3.4.3) COROLLARY (to (3.3)). *Let $\bar{\mathcal{O}}^{\mathrm{norm}}$ be as in* (3.3) *and choose $x \in \bar{\mathcal{O}}^{\mathrm{norm}}$ so that Gx has codimension two. Then the singularity of $\bar{\mathcal{O}}^{\mathrm{norm}}$ in x is smoothly equivalent to a rational double point.*

PROOF. The result follows from the assertions (a)–(c) below.

(a) There exists a transverse slice S in $\bar{\mathcal{O}}^{\mathrm{norm}}$ for x; the dimension of S is two and S has an isolated singularity at x.

(b) Both the property to have a rational singularity at x and the Gorenstein property carry over from $\bar{\mathcal{O}}^{\mathrm{norm}}$ to S.

(c) Rational two-dimensional singularities which are Gorenstein are rational double points.

Let us check the assertions. To obtain (a) we follow a standard procedure of constructing transverse slices (see [S], 5.1, Lemmas 1, 2).

Let $\nu : \bar{\mathcal{O}}^{\mathrm{norm}} \to \mathfrak{g}$ be the composition of the normalization map $\bar{\mathcal{O}}^{\mathrm{norm}} \to \bar{\mathcal{O}}$ with the closed embedding $\bar{\mathcal{O}} \hookrightarrow \mathfrak{g}$. Set $y = \nu(x)$. Let T be a transverse slice for $\mathfrak{g}$ in y. By [S], 5.1, Remark 2, $\dim T = \dim \mathfrak{g} - \dim Gy$, so $\dim(T \cap Gy) = 0$.

Now, a transverse slice S for $\bar{\mathcal{O}}^{\mathrm{norm}}$ in x can be obtained from the cartesian diagram

$$\begin{array}{ccc} S & \longrightarrow & \bar{\mathcal{O}}^{\mathrm{norm}} \\ \big\downarrow & & \big\downarrow \nu \\ T & \longrightarrow & \mathfrak{g} \end{array}$$

(cf. [S], 5.1, Lemma 2). One has obviously $S \cap Gx = \nu^{-1}(T \cap Gy)$. Since ν is finite, $\dim(S \cap Gx) = 0$ and so $\dim S = \dim \bar{\mathcal{O}}^{\mathrm{norm}} - \dim Gx + \dim(S \cap Gx) = 2$.

To check that S is Gorenstein it suffices to note that (S,x) and $(\bar{\mathcal{O}}^{\mathrm{norm}},x)$ are smoothly equivalent and to use the standard theorem on the behaviour of the Gorenstein property under flat morphisms (cf. [Ma], 23.4).

Let us show S has rational singularities. In fact, by théorème 5 of [E] applied to the morphism $\mu : G \times S \to \bar{\mathcal{O}}^{\mathrm{norm}}$, $G \times S$ has rational singularities. Note that the map $\mathrm{pr}_1 : G \times S \to G$ has an obvious simultaneous resolution

$$\begin{array}{ccc} G \times \tilde{S} & \longrightarrow & G \times S \\ \big\downarrow \mathrm{pr}_1 & & \big\downarrow \mathrm{pr}_1 \\ G & = \!= \!= & G \end{array}$$

where $\tilde{S} \to S$ is a desingularization for S. Then théorème 3 of [E] applied to pr_1 gives that S has rational singularities. Finally, for the assertion (c) see [W], 2.5–2.6.

The corollary is proven.

4. Duality theorem: a sketch

We formulate here a part of the duality theorem for coherent sheaves. For more details see [H], VII.3.4.

(4.1) Derived category

Here we fix some notations and recall some standard facts about the derived category of $\mathcal{O}_X$-modules for a scheme X.

Let $\mathfrak{A}$ be an abelian category.

Denote by $K(\mathfrak{A})$ the category whose objects are complexes over $\mathfrak{A}$ and whose morphisms are homotopy classes of morphisms of complexes of degree zero. Denote by $K^+(\mathfrak{A})$ (resp. $K^-(\mathfrak{A})$) the full subcategory of $K(\mathfrak{A})$ consisting of complexes bounded below (resp. above). Set $K^b(\mathfrak{A}) = K^+(\mathfrak{A}) \cap K^-(\mathfrak{A})$.

Let *Qis* denote the set of morphisms of $K(\mathfrak{A})$ which induce an isomorphism in cohomology. The *derived category* of $\mathfrak{A}$, $D(\mathfrak{A})$, is defined as the localization of

$K(\mathfrak{A})$ with respect to Qis. So, $D(\mathfrak{A})$ has the same objects as $K(\mathfrak{A})$. A morphism $f: X \to Y$ in $D(\mathfrak{A})$ is represented by a diagram $X \xleftarrow{s} X' \xrightarrow{g} Y$ in $K(\mathfrak{A})$ with $s \in Qis$.

Let $D^+(\mathfrak{A})$, $D^-(\mathfrak{A})$, $D^b(\mathfrak{A})$ be the full subcategories of $D(\mathfrak{A})$ consisting of complexes with (appropriately) bounded cohomology.

Now let X be a prescheme and let $\mathrm{mod}(\mathcal{O}_X)$ denote the category of $\mathcal{O}_X$-modules. We write $D(X)$ instead of $D(\mathrm{mod}(\mathcal{O}_X))$ and use $D_{qc}(X)$ (resp. $D_c(X)$) to denote the full subcategory of $D(X)$ consisting of complexes with quasi-coherent (resp. coherent) cohomology.

(4.1.1) LEMMA ([H], II.7). *Suppose X is a locally Noetherian prescheme. Then there is a natural equivalence of categories*

$$D^+(Qco(X)) \xrightarrow{\sim} D^+_{qc}(X),$$

where $Qco(X)$ is the category of quasi-coherent sheaves on X.

(4.2) Dualizing complex

The functor $\mathcal{H}om_X : \mathrm{mod}(\mathcal{O}_X)^{op} \times \mathrm{mod}(\mathcal{O}_X) \to \mathrm{mod}(\mathcal{O}_X)$ defines *the right derived functor*

$$\underline{R}\mathcal{H}om_X : D(X)^{op} \times D^+(X) \to D(X)$$

which can be calculated, for instance, using injective resolutions on the second argument.

Analogously, the functor $\otimes$: $\mathrm{mod}(\mathcal{O}_X) \times \mathrm{mod}(\mathcal{O}_X) \to \mathrm{mod}(\mathcal{O}_X)$ defines the *left derived functor* $\otimes^L : D^-(X) \times D^-(X) \to D^-(X)$ which can be calculated using flat resolutions.

For a morphism of schemes $f: X \to Y$ the right derived functor $\underline{R}f_* : D^+(X) \to D^+(Y)$ of the direct image functor, and the left derived functor $\underline{L}f^* : D^-(Y) \to D^-(X)$ of the inverse image functor, are defined. They can be calculated using injective (resp. flat) resolutions.

We shall denote by $R^if_*(M)$, $M \in D^+(X)$, the i-th cohomology of the complex $\underline{R}f_*(M)$. This corresponds to the usual notation when M is represented by a single sheaf.

(4.2.1) LEMMA ([H], V.1.2). *For any $\omega \in D^+(X)$ there is a natural functorial homomorphism*

$$F \to \underline{R}\mathcal{H}om(\underline{R}\mathcal{H}om(F, \omega), \omega)$$

on $D(X)$.

(4.2.2) Definition. Let X be a locally Noetherian prescheme. A complex $\omega \in D^+_{qc}(X)$ is said to have finite injective dimension if there exists an integer n such that for any $\mathcal{O}_X$-module M one has $H^i(\underline{R}\mathcal{H}om_X(M,\omega)) = 0$ for all $i > n$.

(4.2.3) Definition. Let X be a locally Noetherian prescheme. A complex $\omega \in D^+_c(X)$ is said to be *dualizing* if

(i) ω has finite injective dimension,

(ii) the homomorphism of (4.2.1) is an isomorphism for any $F \in D_c(X)$.

A dualizing complex, if it exists, is unique up to tensoring by an invertible sheaf and shifting (cf. [H], V.3.1).

Example. A Noetherian scheme S of finite Krull dimension is Gorenstein if and only if $\mathcal{O}_X$ represents a dualizing complex on S (cf. [H], V.9.1).

(4.3) Duality theorem (cf. [H], VII.3.4).

Recall that we are working in the category of schemes of finite type over **C**.

For any morphism $f: X \to Y$ a functor

$$f^!: D^+_c(Y) \to D^+_c(X)$$

is defined, such that

(i) for any pair of morphisms $f: X \to Y$, $g: Y \to Z$ one has $f^! g^! \approx (gf)^!$,

(ii) if $f: X \to Y$ is smooth of relative dimension n then for any $F \in D^+_c(Y)$

$$f^!(F) \approx \underline{L}f^*(F) \otimes^L \omega_{X/Y}[n]$$

where $\omega_{X/Y}$ is the sheaf of higher relative differentials and $[n]$ means a shift of "n steps left",

(iii) if $f: X \to Y$ is proper, there is a natural "duality isomorphism"

$$\underline{R}f_*(\underline{R}\mathcal{H}om_X(F, f^! G)) \xrightarrow{\sim} \underline{R}\mathcal{H}om_Y(\underline{R}f_* F, G)$$

for all $F \in D^-_{qc}(X)$, $G \in D^+_c(Y)$,

(iv) the functor $f^!$ takes dualizing complexes over Y to dualizing complexes over X.

5. The proof of the key lemma

Let $\pi: X \to Y$ be a proper birational morphism. Suppose that X is smooth and Y is normal. Let $\phi: \mathcal{O}_X \to \omega_X$ be a morphism inducing an isomorphism $\pi_* \phi$. We shall prove this implies that Y is Gorenstein and has rational singularities.

Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & {\searrow}^{p} \quad {}^{q}{\swarrow} & \\ & \mathrm{Spec}(\mathbf{C}) & \end{array}$$

According to (4.3.(i)) $p^!(\mathbf{C}) = \pi^! q^!(\mathbf{C})$. Applying (4.3.(ii)) to $p : X \to \mathrm{Spec}(\mathbf{C})$ one obtains $p^!(\mathbf{C}) \approx \underline{L}p^*(\mathbf{C}) \otimes^L \omega_X[n] = \omega_X[n]$ where $n = \dim X = \dim Y$. So, denoting by ω_Y the complex $q^!(\mathbf{C})[-n] \in D_c^+(Y)$, we have $\pi^!(\omega_Y) = \omega_X$. By (4.3.(iv)) ω_Y is dualizing over Y.

Apply the duality isomorphism (4.3.(iii)) to the morphism π and complexes ω_X, ω_Y. We obtain

$$\text{(5)} \qquad \underline{R}\pi_*(\underline{R}\mathcal{H}om_X(\omega_X, \omega_X)) \approx \underline{R}\mathcal{H}om_Y(\underline{R}\pi_*(\omega_X), \omega_Y).$$

Since $H^i(\underline{R}\mathcal{H}om_X(\omega_X, \omega_X)) = \mathcal{O}_X$ for $i = 0$ and 0 for $i > 0$, the left-hand side of (5) can be identified with $\underline{R}\pi_*(\mathcal{O}_X)$.

By the Grauert–Riemenschneider theorem (cf. [K], Th. 4) $R^i\pi_*(\omega_X) = 0$ for $i > 0$. Since π is proper birational, X is smooth and Y is normal, one has $\pi_*(\mathcal{O}_X) \approx \mathcal{O}_Y$. So by the assumption of the lemma

$$\text{(6)} \qquad \underline{R}\pi_*(\omega_X) \approx \mathcal{O}_Y$$

and (5) can be rewritten as

$$\text{(7)} \qquad \underline{R}\pi_*(\mathcal{O}_X) \approx \omega_Y.$$

Recall we have the morphism $\phi : \mathcal{O}_X \to \omega_X$. This induces the morphism

$$\omega_Y \approx \underline{R}\pi_*(\mathcal{O}_X) \xrightarrow{R\pi_*(\phi)} \underline{R}\pi_*(\omega_X) \approx \mathcal{O}_Y$$

in $D(Y)$, the isomorphisms being taken from (6), (7). Let us denote it by ψ.

Again by the hypothesis of the lemma, $\psi : \omega_Y \to \mathcal{O}_Y$ induces an isomorphism $H^0(\psi)$ in zero cohomology since $R^0\pi_*(\phi) = \pi_*(\phi)$ is an isomorphism.

Now we shall prove all this implies that Y is Gorenstein. If $i : U \to Y$ is an open immersion, $i^*(\omega_Y)$ is dualizing over U (one can apply, for instance, (4.3.(ii)) to make sure that $i^! = \underline{L}i^* = i^*$) and $i^*(\psi)$ induces an isomorphism in zero cohomology. By definition Y being Gorenstein is a local property. Therefore, it suffices to prove the assertion in the case $Y = \mathrm{Spec}(A)$ where A is a commutative Noetherian ring.

According to (4.1.1) we can work in the category of A-modules instead of the category of $\mathcal{O}_Y$-modules.

So we are given a morphism $\psi : \omega_A \to A$ in $D(A)$ which induces an isomorphism $H^0(\psi) : H^0(\omega_A) \xrightarrow{\sim} A$. By definition $\psi : \omega_A \to A$ is represented by a diagram $\omega_A \xleftarrow{s} \Omega \xrightarrow{\Psi} A$ with $s \in Qis$. The morphism $\Psi : \Omega \to A$ of complexes of A-modules is given by the following diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Omega^{-1} & \xrightarrow{d^{-1}} & \Omega^0 & \xrightarrow{d^0} & \Omega^1 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \Psi^0} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

with $\Psi^0 d^{-1} = 0$.

Since $H^0(\Psi) : H^0(\Omega) \to A$ is an isomorphism, there exists a cycle $z \in \Omega^0$ such that $\Psi^0(z) = 1$. Therefore the A-module map $a \mapsto az$ of $A \to \Omega^0$ extends to a morphism $\sigma : A \to \Omega$ of complexes. One has obviously $\Psi\sigma = \mathrm{id}_A$. Thus, $\Omega \approx A \oplus \operatorname{Ker} \Psi$.

Now, Ω has finite injective dimension (it is isomorphic to ω_A in $D(A)$), so A has also finite injective dimension. Therefore A is Gorenstein.

Now, since Y is Gorenstein, ω_Y has an only non-zero cohomology, so $\psi : \omega_Y \to \mathcal{O}_Y$ is an isomorphism.

Then $R^i \pi_*(\mathcal{O}_X) \approx H^i(\omega_Y) = 0$ for $i \neq 0$.

The lemma is proven.

6. Remarks

The remarks below are derived from a private communication of H. Kraft.

(6.1) There is the following direct construction of a G-invariant section of the dualizing sheaf ω_X on X (see (3.2.2)).

Let $\alpha : V \to \wedge^2 \mathfrak{g}_{-1}^* = \wedge^2 \mathfrak{g}_1$ be defined by the formula

$$\alpha(v)(x, y) = \langle [x, y], v \rangle, \qquad x, y \in \mathfrak{g}_{-1}.$$

The alternating form on $\mathfrak{g}_{-1}$ given by $\alpha(e)$ is non-degenerate, so $\dim \mathfrak{g}_1 = 2m$ is even and α induces a non-zero P-equivariant map $\alpha^m : S^m V \to \wedge^{2m} \mathfrak{g}_1$. Then composing α^m with diagonal δ and multiplication μ one obtains

$$\mu : SV^* \xrightarrow{1 \otimes \delta} SV^* \otimes S^m V^* \otimes S^m V \xrightarrow{\mu \otimes 1} SV^* \otimes S^m V \xrightarrow{1 \otimes \alpha^m} SV^* \otimes \wedge^{2m} \mathfrak{g}_1 .$$

The category of $G - \mathcal{O}_X$-modules is equivalent to the category of $P - SV^*$-modules (compare with (3.2)). In this the $G - \mathcal{O}_X$-module $\mathcal{O}_X$ corresponds to the $P - SV^*$-module SV^* and ω_X corresponds to $SV^* \otimes \Lambda^{2m} \mathfrak{g}_1$. Therefore, $\phi : SV^* \to SV^* \otimes \Lambda^{2m} \mathfrak{g}_1$ defines a G-invariant section of ω_X as required.

(6.2) The key lemma (2.3) is a strengthened version of Satz (1.3) in H. Flenner, Archiv der Math. **36** (1981), 35-44. Also Theorem (3.3) can be deduced by proving that all the hypotheses of Flenner's Satz (1.3) are satisfied. This was essentially the route suggested by Panyushev.

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